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CES Working Papers

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2007.78

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://ces.univ-paris1.fr/cesdp/CES-docs.htm
ISSN : 1955-611X

Applications of A Generalized Ky Fan's Matching Theorem In Minimax and Variational Inequality

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Abstract

We present some application of the generalized Ky Fan's Matching Theorem stated by Chebbi, Gourdel and Hammami in minimax and variational inequalities using a generalized coercivity type condition for correspondences defined in L-space.

Key words and phrases: L-structures, L-spaces, L-KKM correspondences, L-coercing family, minimax and variational inequalities.
Classification-JEL: C02, C69, C72.

The purpose of this paper is to give some application of the generalized Ky Fan's Matching theorem stated by Chebbi, Gourdel and Hammami [CGH] to minimax and variational inequalities. All these results extend classical results obtained in topological vector spaces by Fan in [F2] [F3], Ding and Tan in [DT] and Yen in [Y] as well as results obtained in H-spaces by Bardaro and Ceppitelli in [BC1] and [BC2] or in convex spaces in the sense of Lassonde in [L].

In this article, we will use the same notation as in [CGH]. We remind the definition given in [CGH] of L-KKM correspondences, which extend the notion of KKM correspondences to L-spaces, and the concept of L-coercing family for correspondences defined in L-spaces. Let A be a subset of a vector space X. We denote by $\langle A\rangle$ the family of all nonempty finite subsets of A and $convA$ the convex hull of A. Since topological spaces in this paper are not supposed to be Hausdorff, following the terminology used in [B], a set is called *quasi-compact* if it satisfies the Finite Intersection Property while a Hausdorff quasi-compact is called compact. In what follows, the correspondences are represented by capital letters F, G, Q, S, Γ, ... and the single valued functions will be represented by small letters. We denote by $graphF$ the graph of the correspondence F. If X and Y are two topological spaces,

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$\zeta(X, Y)$ denotes the set of all continuous functions from X to Y.

If n is any integer, Δ_n denotes the unit-simplex of $\mathbb{R}^{n+1}$ and for every $J \subset \{0, 1, \ldots, n\}$, Δ_J denotes the face of Δ_n corresponding to J. Let X be a topological space. An *L-structure* (also called *L-convexity*) on X is given by a correspondence $\Gamma : \langle X \rangle \to X$ with nonempty valued such that for every $A = \{x_0, \ldots, x_n\} \in \langle X \rangle$, there exists a continuous function $f^A : \Delta_n \to \Gamma(A)$ such that for all $J \subset \{0, \ldots, n\}$, $f^A(\Delta_J) \subset \Gamma(\{x_j, j \in J\})$. Such a pair (X, Γ) is called an *L-space.* A subset $C \subset X$ is said to be *L-convex* if for every $A \in \langle C \rangle$, $\Gamma(A) \subset C$. A subset $P \subset X$ is said to be *L-quasi-compact* if for every $A \in \langle X \rangle$, there is a quasi-compact L-convex set D such that $A \cup P \subset D$. Clearly, if C exists an L-convex subset of an L-space (X, Γ), then the pair $(C, \Gamma_{|_{\langle C \rangle}})$ is an L-space.

1 A Generalized Ky Fan's Matching Theorem

In this section we remind some known definitions of L-KKM correspondences and L-coercing family quoted in [CGH] and we give a more adapted theorem than the mean result of [CGH] in order to generalize Fan's minimax inequality.

Definition 1.1 *Let (X, Γ) be an L-space and $Z \subset X$ an arbitrary subset. A correspondence $F : Z \to X$ is called L-KKM if and only if:*

$$\forall A \in \langle Z \rangle, \qquad \Gamma(A) \subset \bigcup_{x \in A} F(x).$$

Definition 1.2 *Let Z be an arbitrary set of an L-space (X, Γ), Y a topological space and $s \in \zeta(X, Y)$. A family $\{(C_a, K)\}_{a \in X}$ is said to be L-coercing for a correspondence $F : Z \to Y$ with respect to s if and only if:*

(i) K is a quasi-compact subset of Y,

(ii) for each $A \in \langle Z \rangle$, there exists a quasi-compact L-convex set D^A in X containing A such that:

$$x \in D^A \Rightarrow C_x \cap Z \subset D^A \cap Z,$$

(iii) $\left\{ y \in Y \mid y \in \bigcup_{z \in s^{-1}(y)} \bigcap_{x \in C_z \cap Z} F(x) \right\} \subset K.$

For more explanation of the L-coercivity and to see that this coercivity can't be compared to the coercivity in the sense of Ben-El-Mechaiekh, Chebbi and Florenzano in [BCF], see [CGH].

Definition 1.3 *If X is a topological space, a subset B of X is called strongly compactly closed (open respectively) if for every quasi-compact subset K of X, $B \cap K$ is closed (open, respectively) in K.*

We remind the generalization of Fan's matching theorem of [CGH]:

Theorem 1.1 *Let Z be an arbitrary set in the L-space (X, Γ), Y an arbitrary topological space and $F : Z \to Y$ a correspondence. Suppose that there is a function $s \in \zeta(X, Y)$ such that:*

(a) for every $x \in Z$, $F(x)$ is strongly compactly closed,

(b) the correspondence $R : Z \to X$ defined by $R(x) = s^{-1}(F(x))$ is L-KKM,

(c) t exists an L-coercing family $\{(C_x, K)\}_{x\in X}$ for F with respect to s.

Then $\bigcap\limits_{x\in Z} F(x) \neq \emptyset$, more precisely $K \bigcap (\bigcap\limits_{x\in Z} F(x)) \neq \emptyset$.

For any correspondence $F : X \to Y$, let $F^* : Y \to X$ the "dual" correspondence of F defined, for all $y \in Y$, by $F^*(y) = X \setminus F^{-1}(y)$, where $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$.

The following theorem can be seen as a corollary of Theorem 1.1. It will be used in order to generalizes Fan's minimax inequality.

Theorem 1.2 *Let (X, Γ) an L-space, Y an arbitrary topological space and $F, G : X \to Y$ be two correspondences satisfying:*

(a) for every $x \in X$, $F(x)$ is strongly compactly closed,

(b) for every $x \in X$, $G(x) \subset F(x)$,

(c) there exists a function $s \in \zeta(X, Y)$ such that:

1. *for every $x \in X$, $s(x) \in G(x)$,*
2. *for every $x \in X$, $S^*(x)$ where S is defined by $S(x) = s^{-1}(G(x))$ is L-convex,*
3. *Tthere exists an L-coercing family $\{(C_x, K)\}_{x\in X}$ for F with respect to s.*

Then $\bigcap_{x \in Z} F(x) \neq \emptyset$.

Proof : The correspondence F has strongly compactly closed values and admits an L-coercing family then in order to apply Theorem 1.1, it suffices to show that the correspondence $R : X \to X$ defined by $R(x) = s^{-1}(F(x))$ is L-KKM. Let $A \subset \langle X \rangle$ and $z \in \Gamma(A)$, then by $(c.1)$, $s(z) \in G(\Gamma(A))$. One can check that Condition $(c.2)$ can be equivalently rewritten as $S(\Gamma(A)) \subset S(A)$. Moreover, by $(c.1)$, for all $B \subset X$, $B \subset S(B)$, in particular $\Gamma(A) \subset S(\Gamma(A))$. Hence we deduce, $\Gamma(A) \subset S(A)$. By construction, $S \subset R$, which implies that R is L-KKM. ■

Remark 1.1 *If s is the identity function, the proof of the previous theorem becomes a simple application of Lemma* 1 *of section* 4 *in [H2].*

2 Some Generalizations of Fan's Minimax Inequality

The object of this section is to get a generalization of minimax inequality due to Fan [F3]. In the sequel of this section, for any subset A of $\overline{\mathbb{R}}$ [3] and every $z \in \mathbb{R}$, $A \leq z$ denotes for all $a \in A$, $a \leq z$ and $A \not\leq z$ means that there exists $a \in A$ such that $a > z$.

Definition 2.4 *Let (X, Γ) be an L-space. A correspondence $Q : X \to \overline{\mathbb{R}}$ is said to be weakly lower semi-continuous (weakly l.s.c) on X if for each $p \in \mathbb{R}$, the set $\{x \in X \mid Q(x) \leq p\}$ is closed in X [4] or equivalently, the set $\{x \in X \mid Q(x) \cap]p, +\infty] \neq \emptyset\}$ is open in X.*

Proposition 2.1 *If Q is a lower semi-continuous correspondence then it is weakly lower semi-continuous.*

Proof : The proof is immediate: if for all $p \in \mathbb{R}$, we consider the closed subset $V = \{y \in \mathbb{R} \mid y \leq p\}$, then by l.s.c. $\{x \in X \mid Q(x) \leq p\} = \{x \in X \mid Q(x) \subset V\} = \{x \in X \mid Q(x) \cap V^c = \emptyset\}$ is a closed set.

Let Q be a l.s.c correspondence, we have to prove that for all $p \in \mathbb{R}$, $\{x \in X \mid Q(x) \leq p\}$ is a closed set. For all $p \in \mathbb{R}$, we consider the closed subset $V = \{y \in \mathbb{R} \mid y \leq p\}$ consequently $\{x \in X \mid Q(x) \leq p\} = \{x \in X \mid Q(x) \subset V\} = \{x \in X \mid Q(x) \cap V^c = \emptyset\}$. By the l.s.c. of Q, the set $\{x \in X \mid Q(x) \cap V^c \neq \emptyset\}$ is open then $\{x \in X \mid Q(x) \cap V^c = \emptyset\}$ is closed and the proposition is proved. ■

[3] The extended real line, endowed with its usual topology, see for example Rudin [R]

[4] Recall that a correspondence Q is lower semi-continuous, if for each open set $V \subset Y$, the set $\{x \subset X : Q(x) \cap V \neq \emptyset\}$ is open in X.

Remark 2.2 *Note that the converse implication of Proposition 2.1 is false, since in order to prove that this converse implication is false, we can consider the following counter example: Let the correspondence $Q : \mathbb{R} \to \mathbb{R}$ defined by $Q(x) = \{1, 2\}$ if $x \neq 0$ and $Q(x) = \{-1, 2\}$ if $x = 0$. It is easy to see that Q is weakly l.s.c but not l.s.c.*

We remind a minimax inequality due to Fan [F3].

Theorem 2.3 *(Fan) Let E be a topological vector space, let K be a nonempty compact convex set in E, and let f be a real-valued function on $K \times K$. Suppose that*

(a) for every $y \in X$, $f(y, y) \leq 0$,

(b) for each fixed $y \in K$, the function $x \to f(x, y)$ is quasi-concave on K,

(c) for each fixed $x \in K$, the function $y \to f(x, y)$ is lower semi-continuous on K.

Then there exists a vector y_0 in K such that $f(x, y_0) \leq 0$ for all $x \in K$.

This theorem can be extended in the following way:

Theorem 2.4 *Let (X, Γ) be an L-space and $z \in \mathbb{R}$. Let F and G be two correspondences from $X \times X$ to $\overline{\mathbb{R}}$ satisfying the following condition:*

(a) for every $x \in X$, $G(x, x) \leq z$,

(b) for each fixed $y \in X$, $\{x \in X \mid G(x, y) \not\leq z\}$ is L-convex,

(c) for each fixed $x \in X$, $y \to F(x, y)$ is weakly l.s.c on the quasi-compact subsets of X,

(d) for every $(x, y) \in X \times X$, $F(x, y) \subset G(x, y)$,

(e) there exists a family $\{(C_x, K)\}_{x \in X}$ of pairs of sets satisfying:

(1) K is a quasi-compact subset of X,

(2) for each $A \in \langle X \rangle$, there exists a quasi-compact L-convex set D^A containing A such that:

$$x \in D^A \Rightarrow C_x \subset D^A,$$

(3) $\{y \in X, F(x, y) \leq z$ for all $x \in C_y\} \subset K$.

Then there exists $y_0 \in X$ such that

$$F(x, y_0) \leq z \qquad \forall x \in X.$$

Proof : The technique of the proof follows from the proof of Theorem 1 of Yen [Y], which is based on Fan's lemma [F1][F3], and Theorem 1.2. For each $x \in X$, let $\tilde{F}(x) = \{y \in X : F(x,y) \leq z\}$ and $\tilde{G}(x) = \{y \in X : G(x,y) \leq z\}$. Then by (c), the correspondence $\tilde{F}$ has strongly compactly closed values. By (b), the set $\tilde{G}^*(y) = \{x \in X : G(x,y) \not\leq z\}$ is an L-convex subset of X. By (d), for each $x \in X$, $\tilde{G}(x) \subset \tilde{F}(x)$. Remark that, by (a), for each $x \in X$, $x \in \tilde{G}(x)$ and $\{(C_x, K)\}_{x \in X}$ is an L-coercing family of $\tilde{F}$. Then all the requirements of Theorem 1.2 with s the identity function are satisfied, hence $\bigcap_{x \in X} \tilde{F}(x) \neq \emptyset$ and the theorem is proved. ∎

Remark 2.3 *If we consider the particular case where the correspondence g is a real-valued function in the previous theorem, we can deduce that condition (b) is implied by the classical quasi-concavity of the function $x \to G(x,y)$ for each fixed $y \in X$.*

Remark 2.4 *In view of Remark 2.3, it is easy to see how we can deduce Theorem 2.3 from the previous theorem, it suffices to apply Theorem 2.4 to the correspondences $F = G = f$, $X = K$ which is a nonempty compact convex set in a topological vector space and $z = 0$.*

In the next result, for sake of simplicity, we will focus on the particular case when $F = G$ (but not any more assumed to be a function), and we will weaken conditions (a) and (b) of Theorem 2.4.

Proposition 2.2 *Let (X, Γ) be an L-space, $z \in \mathbb{R}$ and $F : X \times X \to \overline{\mathbb{R}}$ a correspondence satisfying the following condition:*

(a) for each finite subset A of X and for each $y \in \Gamma(A)$, there exists $x_0 \in A$ such that $F(x_0, y) \leq z$,

(b) for each fixed $x \in X$, $y \to F(x,y)$ is weakly l.s.c on quasi-compact subsets of X,

(c) there exists a family $\{(C_x, K)\}_{x \in X}$ of pairs of sets satisfying:

(1) K is a quasi-compact subset of X,

(2) for each $A \in \langle X \rangle$, there exists a quasi-compact L-convex set D^A containing A such that:

$$x \in D^A \Rightarrow C_x \subset D^A,$$

(3) $\{y \in X, F(x,y) \leq z \text{ for all } x \in C_y\} \subset K$.

Then, there exists $y_0 \in X$ such that $F(x, y_0) \leq z$ for all $x \in X$.

Proof : This proof mimics the proof of Fan Inequality: consider the correspondence $S_z : X \to X$ given by $S_z(x) = \{y \in X \mid F(x, y) \nleq z\}$ and assume (arguing by contradiction) that for each $y \in X$ there exists $x \in X$ such that $F(x, y) \nleq z$. Then for each $y \in X$, $S_z^{-1}(y)$ is nonempty. For each fixed $x \in X$, $y \to F(x, y)$ is weakly l.s.c. on the quasi-compact subsets of X then for each fixed $x \in X$, $S_z(x) = \{y \in X \mid F(x, y) \nleq z\}$ is strongly compactly open in X. Consider the correspondence $\tilde{F}_z : X \to X$ given by $\tilde{F}_z(x) = X \setminus S_z(x)$ for $x \in X$. Then $\tilde{F}_z$ is strongly compactly closed in X. It follows from (c) that $\{(C_x, K)\}_{x \in X}$ is an L-coercing family of $\tilde{F}_z$. Indeed let $a \in \tilde{F}_z(x)$ for all $x \in C_a \Rightarrow a \notin S_z(x)$ for all $x \in C_a \Rightarrow F(x, a) \leq z$ for all $x \in C_a \Rightarrow a \in K$. If $\tilde{F}_z$ was L-KKM, by theorem 1.1 with s the identity function, we would have $\bigcap_{x \in X} \tilde{F}_z(x) \neq \emptyset$, in contradiction with condition : $S_z^{-1}(y)$ is nonempty for each $y \in X$. So $\tilde{F}_z$ is not L-KKM and there exists $A \subset \langle X \rangle$ such that $\Gamma(A) \not\subset \bigcup_{x \in A} \tilde{F}_z(x) \Rightarrow \Gamma(A) \not\subset \bigcup_{x \in A} X \setminus S_z(x) \Rightarrow \exists\, y_0 \in \Gamma(A)$ such that $y_0 \notin \bigcup_{x \in A} X \setminus S_z(x) \Rightarrow y_0 \in \bigcap_{x \in A} S_z(x) \Rightarrow y_0 \in S_z(x)$ for all $x \in A$. Then there exists $A \in \langle X \rangle$ and $y_0 \in \Gamma(A)$ such that $F(x, y_0) \nleq z$ for all $x \in A$. Which contradicts condition (a) and the proposition is proved. ∎

Proposition 2.3 *Condition (a) of proposition 2.2 weaken the conditions (a) and (b) of Theorem 2.4.*

Proof : Indeed let us show that Conditions (a) and (b) of Theorem 2.4 imply Condition (a) of Proposition 2.2. Let (X, Γ) be an L-space, $z \in \mathbb{R}$ and F a correspondences from $X \times X$ to $\mathbb{R}$. Let us consider the correspondence $S_z : X \to X$ given by $S_z(y) = \{x \in X \mid F(x, y) \nleq z\}$ and suppose that Condition (b) of Theorem 2.4 hold then for each $y \in X$, $S_z(y)$ is L-convex. Let A be a finite subset of X and $\tilde{y} \in \Gamma(A)$, then $S_z(\tilde{y})$ is an L-convex set. By Assumption (a) of Theorem 2.4, for all $x \in X$, $F(x, x) \leq z$ then $\tilde{y} \notin S_z(\tilde{y})$ and thereby $\Gamma(A) \not\subset S_z(\tilde{y})$. By the L-convexity, $A \not\subset S_z(\tilde{y})$ then there exists $x_0 \in A$ such that $F(x_0, \tilde{y}) \leq z$.

Remark 2.5 *In order to prove that Condition (a) of proposition 2.2 do not imply Conditions (a) and (b) of Theorem 2.4, we can consider the following counter example. Let $X = [0, \pi]$, $z = 0$ and for all $A \in \langle X \rangle$, $\Gamma(A) = co(A)$. The function $f : [0, \pi]^2 \to \mathbb{R}$ given by $f(x, y) = -y \sin(x)$ satisfies condition (a) of proposition 2.2 but f is not quasi-concave in x, (note that f is quasi-convex).*

3 Variational Inequalities

In this section we will prove the existence of solutions of variational inequalities using Theorem 2.4.

Let E and P denote two real topological vector space, X a nonempty convex set in E and $\langle\cdot,\cdot\rangle$ a bilinear form on $P\times E$ whose for each fixed $v\in P$, the restriction of $\langle v,\cdot\rangle$ on any quasi-compact subset Q of X is continuous[5].

Definition 3.5 *A non empty valued correspondence $T: X\to P$ is said to be monotone if for each $(x_1,u_1),(x_2,u_2)\in graphT$ we have $\langle u_1-u_2, x_1-x_2\rangle \geq 0$.*

Theorem 3.5 *Let $T: X\to P$ be a monotone correspondence, $\varphi: X\to\mathbb{R}$ a quasi-convex function lower semi-continuous on any quasi-compact subset of X[6]. Let us suppose that there exists a family $\{(C_x,K)\}_{x\in X}$ of pairs of sets satisfying:*

(a) K is a quasi-compact subset of X,

(b) for each $A\in\langle X\rangle$, there exists a quasi-compact convex set D^A containing A such that:
$$x\in D^A \Rightarrow C_x\subset D^A,$$

(c) $$\left\{y\in X,\ \sup_{u\in T(x)}\{\langle u,y-x\rangle+\varphi(y)-\varphi(x)\}\leq 0 \ for\ all\ x\in C_y\right\}\subset K.$$

Then there is a point $y_0\in X$ such that

$$\sup_{u\in T(x)}\langle u,y_0-x\rangle\leq\varphi(x)-\varphi(y_0),\forall x\in X.$$

Proof : The proof is similar to the proof of Yen [Y]. For each $(x,y)\in X\times X$, let's consider the correspondences F and G defined by

$$G(x,y)=]-\infty,\inf_{v\in T(y)}\{\langle v,y-x\rangle+\varphi(y)-\varphi(x)\}],$$

$$F(x,y)=]-\infty,\sup_{u\in T(x)}\{\langle u,y-x\rangle+\varphi(y)-\varphi(x)\}].$$

The monotonicity of T ensures that for each $(x,y)\in X\times X$, $F(x,y)\subset G(x,y)$. By the quasi-convexity of φ, it follows that for all $p\in\mathbb{R}$, $\{x\in X\mid \varphi(x)<p\}$ is a convex subset of X then for each $y\in X$, $\{x\in X\mid G(x,y)\not\leq$

[5]Which is equivalent, if we denote for all $x\in Z$, $\varphi_v(x)=\langle v,x\rangle$, to : for every closed subset F of $\mathbb{R}$, $\varphi^{-1}(F)$ is a strongly compactly closed subset.

[6]Or equivalently: for every $\alpha\in\mathbb{R}$, $\varphi^{-1}(]-\infty,\alpha])$ is a strongly compactly closed set.

$p\} = \{x \in X \mid p < \inf_{v \in T(y)} \{\langle v, y - x\rangle + \varphi(y) - \varphi(x)\}\}$ is a convex set. Since for each fixed $x \in X$, the function $y \to \sup_{u \in T(x)} \{\langle u, y - x\rangle + \varphi(y) - \varphi(x)\}$ is lower semi-continuous on quasi-compact subsets of X, then F is a weakly l.s.c correspondence on the quasi-compact subsets of X. Consequently, the correspondences F and G are satisfying all the assumptions of Theorem 2.4 with X a convex subset of the topological vector space E and $z = 0$. Hence, there exists $y_0 \in X$ such that $F(x, y_0) \leq 0,\ \forall x \in X$ then

$$\sup_{u \in T(x)} \langle u, y_0 - x\rangle \leq \varphi(x) - \varphi(y_0),\ \forall x \in X.$$

■

Remark 3.6 *In view of the monotony of T, it is easy to show that:*

$$\exists y_0 \textit{ such that } \inf_{u \in T(y_0)} \langle u, y_0 - x\rangle \leq \varphi(x) - \varphi(y_0), \qquad \forall x \in X \tag{1}$$

$$\Downarrow$$

$$\exists y_0 \textit{ such that } \sup_{v \in T(x)} \langle v, y_0 - x\rangle \leq \varphi(x) - \varphi(y_0), \qquad \forall x \in X \tag{2}$$

In the following proposition, we give the sufficient condition in order to get the converse implication:

Proposition 3.4 *If a monotone correspondence $T : X \to P$ satisfies the following condition:*

(a) for each $(x, y) \in X \times X$, the function $h_{xy} : [0, 1] \to \mathbb{R}$ given for $t \in [0, 1]$ by $h_{xy}(t) = \inf_{u \in T((1-t)y+tx)} \langle u, y - x\rangle$ is lower semi-continuous at $t = 0$ (resp. the function $\tilde{h}_{xy} : [0, 1] \to \mathbb{R}$ given for $t \in [0, 1]$ by $\tilde{h}_{xy}(t) = \sup_{u \in T((1-t)y+tx)} \langle u, x - y\rangle$ is upper semi-continuous at $t = 0$),

and the function $\varphi : X \to \mathbb{R}$ is convex then (2) $\Rightarrow$ (1) *in Remark 3.6.*

Proof : Suppose that there exists $y_0 \in X$ such that $\sup_{u \in T(x)} \langle u, y_0 - x\rangle \leq \varphi(x) - \varphi(y_0), \forall x \in X$. For each x' in X, let $x_r = y_0 - r(y_0 - x')$, for all $0 < r < 1$. By the convexity of X, $x_r \in X$ then $\sup_{u \in T(x_r)} \langle u, y_0 - x_r\rangle \leq \varphi(x_r) - \varphi(y_0)$. The convexity of φ implies that $\varphi(x_r) - \varphi(y_0) \leq r(\varphi(x') - \varphi(y_0))$ for all $0 < r < 1$. Hence, $\inf_{u \in T(x_r)} \langle u, y_0 - x'\rangle \leq \varphi(x') - \varphi(y_0)$. If r tends to 0 then by condition (a), we get $\inf_{u \in T(y_0)} \langle u, y_0 - x'\rangle \leq \varphi(x') - \varphi(y_0)$. ■

Remark 3.7 *One check easily that if a correspondence T is upper hemi-continuous in the sense of Cornet [C1] (see for example [C2] and [F]) then condition (a) of proposition 3.4, used by Lassonde [L] in Theorem 2.11., is satisfied*[7]*:*
For any $(x, y) \in X \times X$, the function $h_{xy} : [0,1] \to \mathbb{R}$ defined by or all $t \in [0,1]$, $h_{xy}(t) = \inf_{u \in T((1-t)y+tx)} \langle u, y-x \rangle$ is lower semi-continuous at the point $t = 0$.

In view of Proposition 3.4, the following corollary is deduced from Theorem 3.5.

Corollary 3.1 *Let $T : X \to P$ be a monotone correspondence, $\varphi : X \to \mathbb{R}$ a convex function lower semi-continuous on the quasi-compact subsets of X. Let us suppose that there exists a family $\{(C_x, K)\}_{x \in X}$ of pairs of sets satisfying:*

(a) K is a quasi-compact subset of X,

(b) for each $A \in \langle X \rangle$, there exists a quasi-compact convex set D^A containing A such that:
$$x \in D^A \Rightarrow C_x \subset D^A,$$

(c) $\left\{ y \in X, \sup_{u \in T(x)} \{\langle u, y-x \rangle + \varphi(y) - \varphi(x)\} \leq 0 \text{ for all } x \in C_y \right\} \subset K,$

(d) for each $(x, y) \in X \times X$, the function $h_{xy} : [0,1] \to \mathbb{R}$ given for $t \in [0,1]$ by $h_{xy}(t) = \inf_{u \in T((1-t)y+tx)} \langle u, y-x \rangle$ is l.s.c. at $t = 0$.

Then there exists point $y_0 \in X$ such that
$$\inf_{u \in T(y_0)} \langle u, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \forall x \in X.$$

References

[A] J.P. Aubin, *Mathematical Methods of Game and Economic Theory*, North-Holland, Amsterdam, (1979).

[7] It suffices to consider p equal to the (continuous) linear form $\langle \cdot, y-x \rangle$ in the following definition given by Cornet: a correspondence $F : X \to P$ is said upper hemi-continuous in a point $x_0 \in X$ in the sense of Cornet if for any continuous linear function p, the function $h : x \to \sup_{y \in \varphi(x)} p(y)$ (resp. $\tilde{h} : x \to \inf_{y \in \varphi(x)} p(y)$) is upper semi-continuous (resp. lower semi-continuous) at the point x_0.

[B] N. Bourbaki, *General Topology: Elements of Mathematics*, Chapters 1-4, (1989) Springer.

[BC1] C. Bardaro and R. Ceppitelli, *Some further generalizations of Knaster-Kuratowski-Mazurkiewicz Theorem and Minimax Inequalities*, J. Math. Anal. Appl. **132** (1989), 484-490.

[BC2] C. Bardaro and R. Ceppitelli, *Fixed point theorems and vector valued minimax theorems*, J. Math. Anal. Appl. **146** (1990), 363-373.

[BCFL] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano and J-V. Llinares, *Abstract convexity and fixed points*, J. Math. Anal. Appl **222**(1998), 138-150.

[BCF] H. Ben-El-Mechaiekh, S. Chebbi and M. Florenzano, *A generalized KKMF principle*, J. Math. Anal. Appl. **309** (2005), 583-590.

[C1] B. Cornet, *Fixed point and surjectivity theorems for correspondences; applications*, Ronéotypé Université de Paris-Dauphine (1975).

[C2] B. Cornet, *Thèse de doctorat d'état : Contributions à la théorie mathématique des mécanismes dynamiques d'allocation des ressources*, Université Paris 9 Dauphine, (1981).

[CGH] S. Chebbi, P. Gourdel and H. Hammami, *A Generalization of Fan's Matching Theorem*, Cahiers de la Maison des Sciences Économiques **60** (2006).

[DT] X.P. Ding and K.K. Tan, *On equilibria of non compact generalized games*, J. Math. Anal. Appl. **177** (1993), 226-238.

[F] M. Florenzano, *General Equilibrium Analysis Existence and Optimality Properties of Equilibria*, Kluwer Academic Publishers. Boston/Dordrecht/London (2003).

[F1] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305-310.

[F3] K. Fan, *A minimax inequality and application, in inequalities III*, **Academic Press**(1972).

[F2] K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519-537.

[H2] C. Horvath, *Contractibility and Generalized Convexity*, J. Math. Anal. Appl. **156** (1991), 341-357.

[KKM] D. Knaster, C. Kuratowski and S. Mazurkiewicz, *Ein Beweis des Fixpunktsatsez für n-dimensionale simplexe*, Fundamental Mathematics **XIV** (1929), 132-137.

[L] M. Lassonde, *On the Use of KKM correspondences in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), 151-201.

[PK] S. Park and H. Kim, *Admissible classes of multifunctions on generalized convex spaces*, Proc Coll. Natur. Sci. SNU **18** (1993), 1-21.

[R] W. Rudin, *Real and Complex Analysis*, Third Edition. **1987, 1974, 1966** by McGRAW-HILL International Editions, Mathematics Series.

[Y] C.L. Yen, *A minimax inequality and its applications to variational inequalities*, Pacific Journal Of Mathematics, **Vol 97, No 2** (1981), 477-481.